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Abstract

A *hole* in a graph is an induced cycle with at least 4 vertices. A graph is *even-hole-free* if it does not contain a hole on an even number of vertices. A *pyramid* is a graph made of three chordless paths $P_1 = a \dots b_1$, $P_2 = a \dots b_2$, $P_3 = a \dots b_3$ of length at least 1, two of which have length at least 2, vertex-disjoint except at a, and such that $b_1b_2b_3$ is a triangle and no edges exist between the paths except those of the triangle and the three edges incident with a.

We give a polynomial time algorithm to compute a maximum weighted independent set in a even-hole-free graph that contains no pyramid as an induced subgraph. Our result is based on a decomposition theorem and on bounding the number of minimal separators. All our results hold for a slightly larger class of graphs, the class of (square, prism, pyramid, theta, even wheel)-free graphs.

1 Introduction

In this article, graphs are finite and simple. A *hole* in a graph is an induced cycle with at least 4 vertices. The *length* of a hole is the number of vertices in it. A graph G *contains* a graph H if some induced subgraph of G is isomorphic to H. A graph G is H*-free* if it does not contain H. When $\mathcal{H}$ is a set of graphs, G is $\mathcal{H}$*-free* if it is H-free for all H in $\mathcal{H}$.

The class of even-hole-free graphs was the object of much research (see [10] for a survey). However, the complexity of computing a maximum independent set in an even-hole-free graph is not known.

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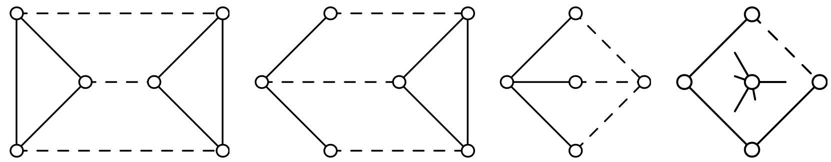

Figure 1: Prism, pyramid, theta and wheel (dashed lines represent paths)

A *pyramid* is a graph made of three chordless paths $P_1 = a \dots b_1$, $P_2 = a \dots b_2$, $P_3 = a \dots b_3$ of length at least 1, two of which have length at least 2, vertex-disjoint except at a, and such that $b_1b_2b_3$ is a triangle and no edges exist between the paths except those of the triangle and the three edges incident with a. See Fig. 1.

Our main result is a polynomial time algorithm to compute a maximum weighted independent set in an (even-hole, pyramid)-free graph. Our approach is by first proving a decomposition theorem for the class of (even-hole, pyramid)-free graph. This theorem might have other applications because the presence of a pyramid in an even-hole-free graphs places significant restrictions on its structure. The graphs seems to "organize itself" around the pyramid, in a way that can likely be exploited algorithmically. Results in this direction appear in Chudnovsky and Seymour [6]. So our result on the pyramid-free case might help to understand the full class of even-hole-free graphs. We use our decomposition theorem to prove that (even-hole, pyramid)-free graph contain polynomially many minimal separators (to be defined in the next section). In fact, we prove this property for a slightly larger class of graphs, namely the (theta, pyramid, prism, even wheel, square)-free graphs. And as we explain in the next section, this property implies the existence of a polynomial time algorithm to compute maximum weighted independent sets.

In section 2 we state formally our main results and motivate them further. In section 3, we prove the decomposition theorem. In section 4, we give several properties of minimal separators in our class of graphs. In section 5, we prove that graphs in our class contain polynomially many minimal separators.

Notation

Let G be a graph. By a *path* we mean a chordless (or induced) path. When P is a path in G, we denote by P^* the path induced by the internal vertices of P. When a and b are vertices of a path P, we denote by aPb the subpath of P with ends a and b. A *clique* in a graph is a set of pairwise adjacent vertices.

When $A, B \subseteq V(G)$, we denote by $N_B(A)$ the set of vertices of $B \setminus A$ that have at least one neighbor in A and $N(A)$ means $N_{V(G)}(A)$. Note that $N_B(A)$ is disjoint from A. We write $N(a)$ instead of $N(\{a\})$ and $N[a]$ for $\{a\} \cup N(a)$. We denote by $G[A]$ the subgraph of G induced by A. To avoid too heavy notation, since there is no risk of confusion, when H is an induced subgraph of G, we write N_H instead of $N_{V(H)}$.

A vertex x is *complete* (resp. *anticomplete*) to A if $x \notin A$ and x is adjacent to all vertices of A (resp. to no vertex of A). We say that A is *complete* (resp. *anticomplete*) to B if every vertex of A is complete (resp. anticomplete) to B (note that this means in particular that A and B are disjoint).

2 Results

Let G be a graph and $a, b \in V(G)$. A set $C \subseteq V(G)$ is an *minimal* (a, b)*-separator* if a and b are in disctint components of $G \setminus C$ and C is minimal with this property. We say that C is a *minimal separator* if C is a minimal (a, b)-separator for some pair a, b.

It is easy to check that a minimal separator in a graph G can be equivalently defined as a set $C \subseteq V(G)$ such that $G \setminus C$ has a connected component L and a connected component R such that every vertex of C has neighbors in both L and R. Note that $G \setminus C$ has possibly more connected components.

Say that a class $\mathcal{C}$ of graphs has the *polynomial separator property* if there exists $b_{\mathcal{C}}$ such that every graph G in $\mathcal{C}$ has at most $|V(G)|^{b_{\mathcal{C}}}$ minimal separators. As explained by Chudnovsky, Pilipczuk, Pilipczuk and Thomassé in [5] (see also the end of Section 5), it follows from results of Bouchité and Todinca [3, 4] that for any class of graphs, having the polynomial separator property implies that the Maximum Weighted Independent Set Problem can be solved in polynomial time.

We are therefore interested in finding classes of graphs where the number of minimal separators is bounded by some polynomial. To gain insight on this question, let us survey examples of graphs with exponentially many minimal separators.

For an integer $k \geq 1$, the *k-prism* is the graph consisting of two cliques on k vertices, and a k-edge matching between them. More precisely, the k-prism G has vertex set $\{a_1, \dots, a_k, b_1, \dots, b_k\}$, each of the sets $\{a_1, \dots, a_k\}$ and $\{b_1, \dots, b_k\}$ is a clique, $a_ib_i \in E(G)$ for every $i \in \{1, \dots, k\}$, and there are no other edges in G. See Fig. 2. As observed in [5], it is easy to check that a k-prism has $2^k - 2$ minimal separators. This suggests that not containing a big matching plays a role in bounding the number of minimal separators, and indeed a simple theorem can be proved in this direction. Call *k-semi-induced matching* any graph whose vertex set can be partitoned into two sets $X = \{x_1, \dots, x_k\}$ and $Y = \{y_1, \dots, y_k\}$ such that the only edges between X and Y are the edges x_iy_i $(i = 1, \dots, k)$. The edges among vertices of X and vertices of Y are unrestricted.

Theorem 2.1 *For every k, every graph G on n vertices that contains no k-semi-induced matching has at most $O(n^{2k-2})$ minimal separators that can be enumerated in time $O(n^{2k})$.*

Proof. Let a and b be two non-adjacent vertices in a graph G that does not contain a k-semi-induced matching, and let C be a minimal separator separating them. Call A and B the components of $G \setminus C$ that contain a and b respectively. By minimality of C, every vertex in C has a neighbor in A. It is therefore well defined to consider an inclusion-wise

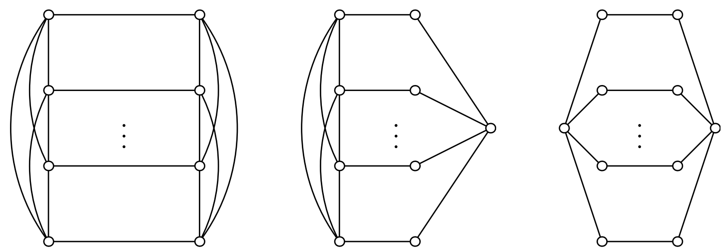

Figure 2: k-prism, k-pyramid, k-theta

minimal subset X_A of A such that $C \subseteq N(X_A)$. For every $x \in X_A$, there exists a vertex $c \in C$ such that $xc \in E(G)$ and no other vertex of X_A is adjacent to c. For otherwise, $X_A \setminus \{x\}$ would contradict the minimality of X_A. It follows that $G[X_A \cup C]$ contains an $|X_A|$-semi-induced matching, so $|X_A| < k$. We may define a similar set X_B, and we observe that $C = N(X_A) \cap N(X_B)$.

From the previous paragraph, the following algorithm enumerates all minimal separators of G: for every pair of sets X_A, X_B of cardinality less than k, compute $C = N(X_A) \cap N(X_B)$ and check whether C is a minimal separator. Since $\binom{n}{i} \leq n^i$, we have $\binom{n}{0} + \cdots + \binom{n}{k-1} \leq kn^{k-1}$, so the algorithm enumerates at most $O(n^{2k-2})$ minimal separators in time $O(n^{2k})$. $\square$

Results in other directions can be proved. Chudnovsky, Pilipczuk, Pilipczuk, and Thomassé [5] proved a graph G that contains no k-prism and no hole of length at least 5 has at most $|V(G)|^{k+2}$ minimal separators. But since we are interested in even-hole-free graphs, we do not want to exclude odd holes.

In Fig. 2, variants of k-prims are shown. They are obtained from k-prisms by subdividing the matching edges (once or twice) and contracting one or two of the cliques into a single vertex. We call these graphs k-pyramids and k-thetas. They are all easily checked to contain exponentially many minimal separators, and we do not define them more formally.

From these three examples, we can see that the so-called *3-path configurations* are maybe important to understand minimal separators. They are defined as being the pyramids (that we already know) and the thetas and prisms that we define now (see Fig. 1).

A *theta* is a graph made of three internally vertex-disjoint chordless paths $P_1 = a \dots b$, $P_2 = a \dots b$, $P_3 = a \dots b$ of length at least 2 and such that no edges exist between the paths except the three edges incident with a and the three edges incident with b.

A *prism* is a graph made of three vertex-disjoint chordless paths $P_1 = a_1 \dots b_1$, $P_2 = a_2 \dots b_2$, $P_3 = a_3 \dots b_3$ of length at least 1, such that $a_1a_2a_3$ and $b_1b_2b_3$ are triangles and no edges exist between the paths except those of the two triangles.

The examples of graphs with exponentially many separators that we have shown so far all contain a theta, a pyramid or a prism. But excluding them is not enough to guaranty

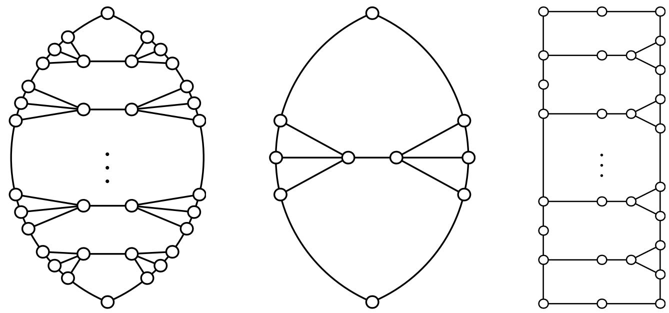

Figure 3: k-turtle, turtle and k-ladder

a polynomial number of minimum separators. In Fig. 3 a construction of graphs with no theta, no pyramid and no prism with exponentially many minimal separator is shown. We call this construction a *k-turtle*, and again it is easy to check that a k-turtle contains exponentially many minimal separators.

The k-turtles suggest defining a *turtle* as a graph made of two internally vertex-disjoint chordless paths $P_1 = u \dots v$, $P_2 = u \dots v$ that form a hole. Moreover, there are two adjacent vertices x, y not in the paths, such that x has at least three neighbors in P_1 (and none in P_2) and y has at least three neighbors in P_2 (and none in P_1), see Fig. 3. A k-turtle contains a turtle, in the same way as a k-prism contains a prism, a k-pyramid contains a pyramid and a a k-theta contains a theta. In Fig. 3 is also represented a construction that we call the k-ladder, that provides examples of even-hole-free graphs with maximum degree 3 and exponentially many minimal separators.

Since we are not able to imagine examples of graphs with exponentially many minimal separators containing no prism, pyramid, theta or turtle, we propose the following conjecture, that would be in some sense the best possible statement regarding bounding the number of minimal separators.

Conjecture 2.2 *There is a polynomial P such that every graph G that contains no prism, pyramid, theta or turtle has at most $P(|V(G)|)$ minimal separators.*

Since we are interested in even-hole-free graphs, it is worth observing that every prism, theta and turtle contains an even hole. For theta and prism, this is because at least two of the three paths must have the same parity and therefore form an even hole. For turtles, it is because every turtle contains an even wheel. Let us define them.

A *wheel* is a graph made of a hole H called the *rim* and a vertex v called the *center* that has at least three neighbors in H (see Fig. 1). An *even wheel* is a wheel whose center has an even number of neighbors in the rim. It is easy to check that every turtle contains an even wheel, and that every even wheel contain an even hole.

A weakening of Conjecture 2.2 is therefore obtained by restricting it to (prism, pyramid, theta, even wheel)-free graphs. Note that (prism, theta, even wheel)-free graphs have been studied under the name of *odd-signable graphs* and they seem to capture essential properties of even-hole-free graphs, for more about them see the survey of Vušković [10]. Interestingly, prisms, pyramids, thetas and wheels are called *Truemper configurations* and they play an important role in many decomposition theorems for classes of graphs, see [11] for a survey. But we were not able to prove that (prism, pyramid, theta, even wheel)-free graphs have polynomially many minimal separators. However, we can prove that if we also exclude *squares* (holes of length 4), then the number of minimal separators is polynomially bounded.

We call $\mathcal{C}$ the class of (square, prism, pyramid, theta, even wheel)-free graphs. Observe that $\mathcal{C}$ is a superclass of the class of (even hole, pyramid)-free graphs. Here is our main result (proved in Section 5).

Theorem 2.3 *Every graph in $\mathcal{C}$ on n vertices contains at most $O(n^8)$ minimal separators. There is an algorithm of complexity $O(n^{10})$ that enumerates them. Consequently, there exists a polynomial time algorithm for the Maximum Weighted Independent Set restricted to $\mathcal{C}$.*

To prove Theorem 2.3, we rely on a decomposition theorem for $\mathcal{C}$. To state it, we need terminology. When H is a hole in some graph and u is a vertex not in H with at least two neighbors in H, we call *u-sector* of H any path of H of length at least 1, whose ends are adjacent to u and whose internal vertices are not. Observe that H is edgewise partitioned into its u-sectors. Let H be a hole in a graph and let u be a vertex not in H. We say that u is *major* w.r.t. H if $N_H(u)$ is not included in a 3-vertex path of H. The decomposition theorem is the following (proved in Section 3).

Theorem 2.4 *Let G be a graph in $\mathcal{C}$, H a hole in G and w a major vertex w.r.t. H. If C is a connected component of $G \setminus N[w]$, then there exists a w-sector $P = x \dots y$ of H such that $N(C) \subseteq \{x, y\} \cup (N(w) \setminus V(H))$.*

Lower bounds for the number of minimal separators in $\mathcal{C}$

For every integer k, there exits a graph in $\mathcal{C}$ with at least $O(k^2)$ minimal separators, and this is the best lower bound that we have so far. A simple example of this phenomenon is a chordless cycle of length $k \geq 5$ (any pair of non-adjacent vertices is a minimal separator). Another example G_k is maybe worth mentioning because it does not contain holes of length greater than 5. Let us describe G_k. Consider four cliques X, Y, X' and Y', each on k vertices. Set $X = \{x_1, \dots, x_k\}$, $Y = \{y_1, \dots, y_k\}$, $X' = \{x'_1, \dots, x'_k\}$, $Y' = \{y'_1, \dots, y'_k\}$. Add a vertex z. Add all possible edges between z and $X \cup X'$. Add all possible edges between Y and Y'. For every $i = 1, \dots, k$, add all possible edges from x_i to

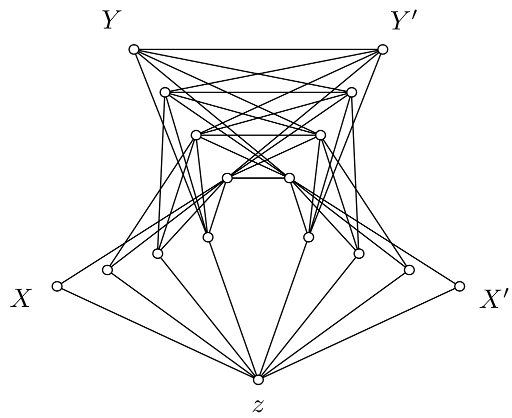

Figure 4: Graph G_4 (edges of the cliques X, Y, X' and Y' are not represented)

$\{y_{k-i+1}, y_{k-i+2}, \dots, y_k\}$ and all possible edges from x'_i to $\{y'_{k-i+1}, y'_{k-i+2}, \dots, y'_k\}$. These are all the vertices and edges of G_k, see Fig. 4 where G_4 is represented.

It is straightforward to check that $G_k \in \mathcal{C}$. To do so, it is convenient to note that every hole H in G_k must go through z and contains exactly one vertex in each of the sets X, X', Y and Y'. So every hole in G_k has length 5. Since squares, even wheels, thetas and prisms all contain even holes, the only obstruction that may exist in G_k is the pyramid. But G_k cannot contain it since in a pyramid there exists a vertex whose neighborhood contains three non-adjacent vertices, and this does not exist in G_k.

For every $i \in \{1, \dots, k\}$, set $C_i = \{x_i, x_{i+1}, \dots, x_k\} \cup \{y_{k-i+2}, y_{k-i+3}, \dots, y_k\}$. Note that for $i \in \{1, \dots, k\}$, $y_1 \notin C_i$ and $C_1 \cap Y = \emptyset$. For every $j \in \{1, \dots, k\}$, set $C'_j = \{x'_j, x'_{j+1}, \dots, x'_k\} \cup \{y'_{k-j+2}, y'_{k-j+3}, \dots, y'_k\}$. It is now easy to check that $C_i \cup C'_j$ is a minimal separator (separating z from y_1) for every pair (i, j) in $\{1, \dots, k\}^2$. Since $|V(G_k)| = 4k + 1$ and there are k^2 pairs (i, j) in $\{1, \dots, k\}^2$, G_k has $O(k^2)$ minimal separators.

Rankwidth and semi-induced matchings in $\mathcal{C}$

One may suspect that graphs in $\mathcal{C}$ are very "simple" in some sense (in which case our result would be less interesting). And it is not so easy to exhibit graphs from $\mathcal{C}$ that are "complex", so it is worth explaining here how to build such graphs. To measure the complexity of a graph we use the notion of rankwidth, that is equivalent to the notion of cliquewidth, in the sense that a class of graphs has unbounded rankwidth if and only if it has unbounded cliquewidth (see [7] for more about cliquewidth).

To provide graphs in $\mathcal{C}$ of arbitrarily large rankwidth, it suffices to note that every hole-free graph (better known as *chordal graphs*) is in $\mathcal{C}$. Chordal graphs are known to have unbounded rankwidth. But chordal graphs are in some sense "simple": they are all complete graphs or have clique separators, and many problems can be solved in polynomial time for them (see [11] for more about that).

In [1] Adler, Le, Müller, Radovanović, Trotignon and Vušković describe even-hole-free graphs of arbitrarily large rankwidth. They are also diamond-free (where the *diamond* is the graph on vertices a, b, c, d with all possible edges except ab) and they have no clique separator. So, to the best of our knowledge, they are "complex". The only problem is that they are not in $\mathcal{C}$ because they contain pyramids. We now explain how to modify graphs defined in [1] to obtain graphs in $\mathcal{C}$.

Graphs defined in [1] all vertex-wise partition into a path P and a clique K. An example is represented in Fig. 5, where $|K| = 4$ and P is represented as a circle around K. Every vertex of K has neighbors in P and every vertex of P has at most one neighbor in K. These graphs contain pyramids that are built as follows: take three vertices a, b, c of K that induce a triangle, and consider neighbors a', b', c' of a, b, c respectively in P. Suppose that a', b', c' are chosen such that $a'Pc'$ has no neighbor of a and c in its interior and b' is the unique neighbor of b in it. The pyramid is then formed by $a'Pc'$ and abc.

To avoid pyramids in graphs from [1], take such a graph G and apply the following algorithm to it:

While there exist a 6-tuple (a, b, c, a', b', c') as above, denote by x and y the two neighbors of b' in P. Remove b' from P, replace it by a path $xp_1p_2p_3p_4p_5p_6p_7y$ and add the following edges: bp_1, bp_4 and bp_7. Note that the obtained graph is still vertex-wise partitioned into a clique and a path, so that our procedure can be applied repeatedly.

Each time, the number of pyramids in the graph decreases so that the algorithm terminates. From the proofs in [1], it is easy to check that we obtain graphs in $\mathcal{C}$ that have unbounded rankwidth. We omit further details that can be found in [1].

We observe that graphs from [1] may contain arbitrarily large semi-induced matchings, so that our main result cannot be a simple corollary of Theorem 2.1.

3 Decomposing graphs in $\mathcal{C}$

Recall that when H is a hole in some graph and u is a vertex not in H with at least two neighbors in H, we call *u-sector* of H any path of H of length at least 1, whose ends are adjacent to u and whose internal vertices are not. Observe that H is edgewise partitioned into its u-sectors.

Let H be a hole in a graph and let u be a vertex not in H. We say that u is *major* w.r.t. H if $N_H(u)$ is not included in a 3-vertex path of H. We omit "w.r.t. H" when H is clear from the context.

Lemma 3.1 *In every graph in $\mathcal{C}$, every major vertex u w.r.t. a hole H has at least five neighbors in H or has exactly three neighbors in H that are pairwise non-adjacent.*

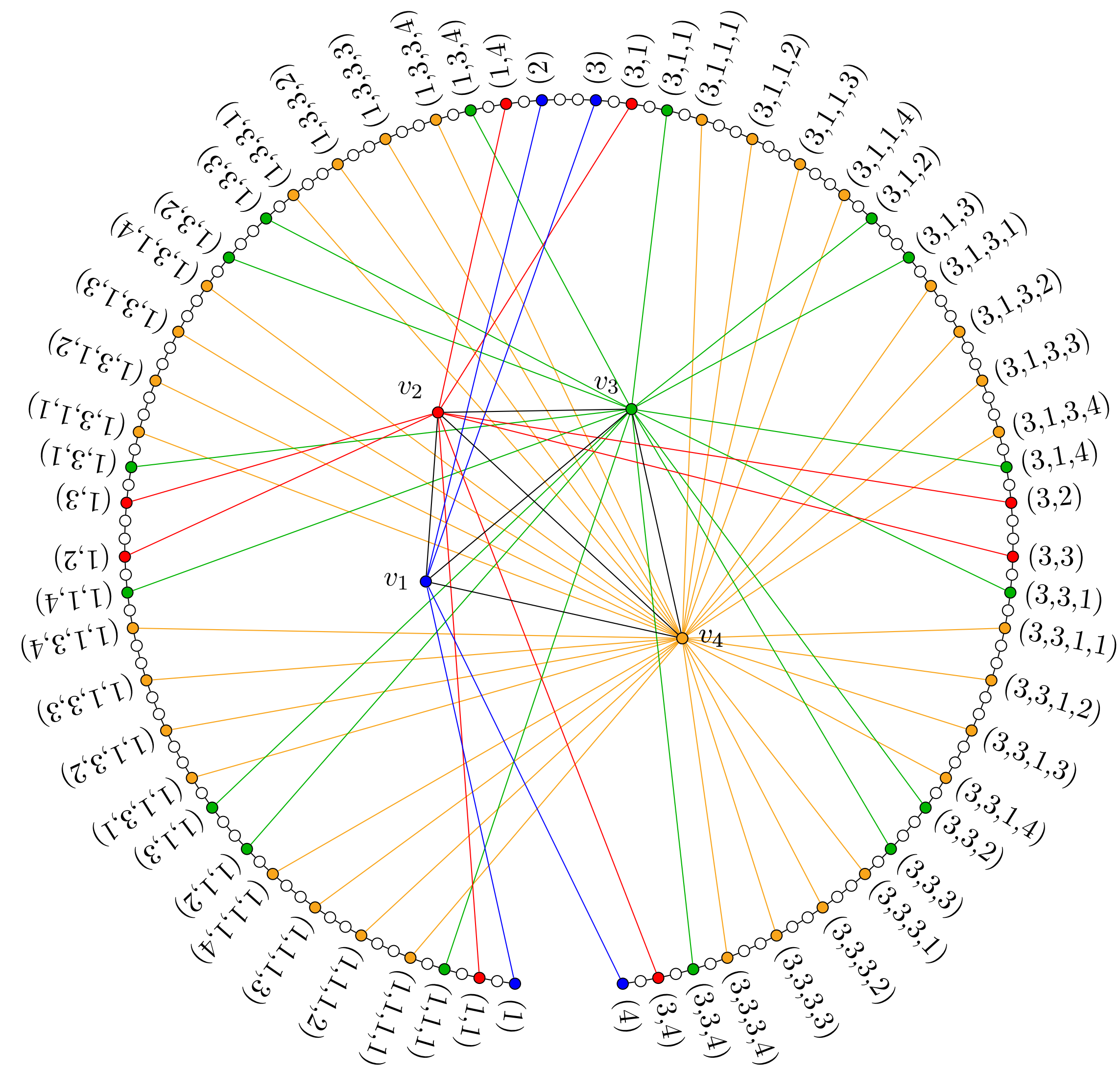

Figure 5: A graph from [1] (the clique contains 4 vertices)

Proof. If u has exactly two neighbors in H, since these two neighbors are not included in a 3-vertex path, they are non-adjacent. Hence, u and H form a theta, a contradiction. If u has exactly three neighbors in H, since these are not included a 3-vertex path, they are pairwise non-adjacent for otherwise u and H form a pyramid. If u has exactly four neighbors in H, then u and H form an even wheel, a contradiction. □

It follows from Lemma 3.1 that if u is major w.r.t. a hole H, then (H, u) is a wheel. We will use this fact throughout the paper. A vertex that is not major with respect to some hole H and still has neighbors in H is *minor* w.r.t. H.

Lemma 3.2 *In a graph from $\mathcal{C}$, every minor vertex u w.r.t. a hole H satisfies one of the following.*

- *u has a unique neighbor in H (we then say that u is* pending *w.r.t. H).*
- *u has two neighbors in H which are adjacent (we then say that u is a* cap *w.r.t. H).*
- *u has three neighbors in H which induce a path xyz (we then say that u is a* clone of *y w.r.t. H).*

Proof. Otherwise, u has two non-adjacent neighbors in H, so u and H form a theta. □

When H is hole and u a clone of y w.r.t. H, we denote by $H_{u\backslash y}$ the hole induced by $\{u\} \cup V(H) \setminus \{y\}$. Observe that y is a clone of u w.r.t. $H_{u\backslash y}$.

Lemma 3.3 *Let H be a hole in a graph $G \in \mathcal{C}$ and u be a clone of y w.r.t. H. Let v be a major vertex w.r.t. H. Then, $vu \in E(G)$ if and only if $vy \in E(G)$. In particular, a vertex is major w.r.t. H if and only if it is major w.r.t. $H_{u\backslash y}$.*

Proof. Suppose that v is adjacent to exactly one of u, y. Since v is major w.r.t. H, (H, v) is a wheel. If $(H_{u\backslash y}, v)$ is also a wheel, then one of (H, v), $(H_{u\backslash y}, v)$ is an even wheel, a contradiction. So, v has exactly two neighbors in $H_{u\backslash y}$, and hence exactly three neighbors in H. By Lemma 3.1 the neighbors of v in H are non-adjacent, but by Lemma 3.2, the neighbors of v in $H_{u\backslash y}$ are adjacent, a contradiction. □

Lemma 3.4 *Let u and v be two non-adjacent major vertices w.r.t. a hole H of a graph $G \in \mathcal{C}$. Let $P = u' \dots u''$ be a u-sector of H. Then one of the following holds.*

(i) P contains at most one neighbor of v, and if it has one, it is either u' or u''.

(ii) $u'u'' \in E(G)$ and v is adjacent to both u' and u''.

(iii) P contains at least 3 neighbors of v.

Proof. Let $R = x \dots y$ be the path induced by $V(H) \setminus V(P)$, with ends such that $u'x \in E(G)$ and $u''y \in E(G)$.

(1) u has a neighbor in the interior of R (in particular, R has length at least 2).

Otherwise, $N_H(u) \subseteq \{u', u'', x, y\}$, contradicting Lemma 3.1. This proves (1).

Suppose first that P contains exactly one neighbor v' of v. Suppose for a contradiction that v' is not an end of P. If $vx \in E(G)$ then $u'v' \notin E(G)$ because G is square-free. Hence, $V(P) \cup \{u, v, x\}$ induces a theta (if $ux \notin E(G)$) or a pyramid (if $ux \in E(G)$). So, $vx \notin E(G)$. Symmetrically, $vy \notin E(G)$. Hence, G contains a theta from u to v': two paths use vertices of P, and the third one goes through v, some neighbor of v in the interior of R (which exists since v is major) and some neighbor of u in the interior of R, which exists by (1). So, (i) holds.

Suppose now that P contains exactly two neighbors v' and v'' of v. If $u'u'' \in E(G)$, then (ii) holds, so we may assume that $u'u'' \notin E(G)$. Hence, u and P form a hole that we denote by H_u. We have $v'v'' \in E(G)$ for otherwise, v and H_u form a theta. By Lemma 3.1, v has at least five neighbors in H, so at least one of them is in the interior of R. Also, u has a neighbor in the interior of R by (1). Hence, H_u together with a shortest path from u to v with interior in the interior of R form a pyramid, a contradiction.

Finally, if P contains at least three neighbors of v, then (iii) holds. $\square$

Let H be a hole in a graph and let u and v be two vertice not in H. We say that u and v are *nested* w.r.t. H if H contains two distinct vertices a and b such that one (a, b)-path of H contains all neighbors of u in H, and the other one contains all neighbors of v in H. Observe that u and v may both be adjacent to a or to b. Observe that under the assumption that v has at least two neighbors in H (so that the notion of v-sector is defined), u and v are nested if and only if there exists a v-sector that contains every neighbor of u in H. If u is a cap, a pending vertex, or a vertex with no neighbor in H, then it is nested with all other vertices not in H.

Lemma 3.5 *Let H be a hole in a graph $G \in \mathcal{C}$. If u and v are major or clones w.r.t. H and are nested, then $uv \notin E(G)$.*

Proof. Otherwise, let H_u be the hole formed by u and the u-sector of H that contains all neighbors of v. Then one of (H, v) or (H_u, v) is an even wheel, a contradiction. $\square$

If u and v are two vertices not in H and not nested w.r.t. H, then they *cross* on H.

Lemma 3.6 *Let H be a hole in a graph $G \in \mathcal{C}$ and let u and v be two vertice not in H. If u and v cross, then one the following holds.*

(i) H contains four vertices u', u'', v' and v'' such that:

- *u', v', u'' and v'' are distinct and appear in this order along H;*

- *$u', u'' \in N(u)$;*
- *$v', v'' \in N(v)$.*

(ii) $N_H(u) = N_H(v)$, $N_H(u)$ is an independent set and $|N_H(u)| = 3$.

(iii) $N_H(u) = N_H(v)$ and both u and v are clones w.r.t. H.

Proof. Since a vertex with no neighbor in H, a cap or a vertex pending w.r.t. H is nested with any other vertex not in H, by Lemma 3.2, u and v are major or clones w.r.t. H. Hence, consider two non-adjacent neighbors a, b of u in H. Since v is major or clone, v has a neighbor in the interior of one (a, b)-path P_v of H. We suppose that a, b and P_v are chosen subject to these properties ($ab \notin E(G)$, v has a neighbor in the interior of P_v) and so that P_v is minimal. If v has neighbors in the interior of the other (a, b)-path of H, then (i) holds.

Otherwise, $N_H(v) \subseteq V(P_v)$. If P_v is a u-sector, then u and v are nested, so suppose that u has a neighbor u' that is an internal vertex of P_v. By the minimality of P_v, $N_H(v) \subseteq \{a, b, u'\}$. Hence, either v is a clone of u' w.r.t. H, or by Lemma 3.1 applied to v, $N_H(v) = \{a, b, u'\}$ and $N_H(v)$ is an independent set. So, if $N_H(u) \geq 4$, then (i) holds, and if $N_H(u) = 3$, then (ii) or (iii) holds. $\square$

Lemma 3.7 *Let H be a hole in a graph $G \in \mathcal{C}$ and suppose that u and v are two major vertices w.r.t. H. Then $uv \in E(G)$ if and only if u and v cross.*

Proof. If u and v are nested, then $uv \notin E(G)$ by Lemma 3.5. It remains to prove the converse: if u and v cross, then they are adjacent. So suppose for a contradiction that they are not adjacent.

We apply Lemma 3.6. Since u and v are major, (iii) does not hold. If (ii) holds, then G contains a square, a contradiction. Hence we may assume that (i) holds: there exist in H two neighbors u', u'' of u and two neighbors v', v'' of v such that u', v', u'', v'' are distinct and appear in this order along H. We choose them so that the path P from u' to u'' in $H \setminus v''$ is minimal. We now break into two cases.

Case 1: P is not a u-sector.

So, let u''' be a neighbor of u in the interior of P. By the minimality of P, $u'Pu'''$ and $u'''Pu''$ are u-sectors and have no neighbor of v in their interior, so $v' = u'''$. Our goal in this case is to show the existence of three paths R_1, R_2 and R_3 forming a theta from u to v. We set $R_1 = uv'v$.

If v is adjacent to both u' and u'', then $\{u, v, u', v'\}$ induces a square, a contradiction. So we may assume up to symmetry that v is not adjacent to u''. W.l.o.g. we may assume that v'' is such that $Q = v' \dots v''$ is a v-sector of H (that contains u''). By Lemma 3.4, Q contains at least three neighbors of u. So, there exists a path R_2 from u to v going through v'' and the interior of R_2 contains no neighbors of v'.

Let x be the neighbor of v'' in H that is not in Q and let R be the path of H from x to u' that does not contain P.

We claim that v has a neighbor in the interior of R (which therefore has length at least 2). Otherwise, $N_H(v) \subseteq \{u', v', v'', x\}$. By Lemma 3.1, $N_H(v) = \{u', v', v''\}$ and $u'v' \notin E(G)$. So, $\{u, v, u', v'\}$ induces a square, a contradiction.

We claim that u has a neighbor in the interior of R. For suppose not. Since G contains no even wheel, u has an odd number of neighbors in Q, and since it also has an odd number of neighbors in H, u must be adjacent to x. Let $Q' = y \dots x$ be the u sector of H that contains v''. Since G is square-free, v is not adjacent to x or y. Hence, Q' contains a unique neighbor of v, that is in its interior, a contradiction to Lemma 3.4.

Now, by considering a path R_3 from u to v with interior in the interior of R (which exists from the two claims we just proved), we see that R_1, R_2 and R_3 form a theta.

Case 2: P is a u-sector.

We apply Lemma 3.4 to P and we observe that outcomes (i) and (ii) do not hold, so outcome (iii) holds: P contains at least three neighbors of v. It follows that there exist two internally vertex disjoint paths R_1 and R_2, both from u to v, with interior in P and such that $V(R_1) \cup V(R_2)$ induces a hole. Let x be the neighbor of u' in H that is not in P, and y be the neighbor of u'' in H that is not in P. Let R be the path of H from x to y that does not contain P.

We claim that u has a neighbor in the interior of R. Otherwise, $N_H(u) \subseteq \{u', u'', x, y\}$, contradicting Lemma 3.1.

We claim that v has a neighbor in the interior of R. Otherwise, v'' must be one of x or y, say x up to symmetry. But since there is no even wheel in G, v has an odd number of neighbors in P and in H, so v must be adjacent to y. Since G is square-free, v cannot be adjacent to both u', u'', so suppose up to symmetry that is not adjacent to u''. Hence, u'' is the unique neighbor of u in some v-sector of H (moreover in its interior), a contradiction to Lemma 3.4.

Now, by considering a path R_3 from u to v with interior in the interior of R (which exists from the two claims we just proved), we see that R_1, R_2 and R_3 form a theta, a contradiction. $\square$

Lemma 3.8 *Let H be a hole in a graph $G \in \mathcal{C}$. If u and v are non-adjacent vertices of $G \setminus H$ that cross, then u and v are both clones w.r.t. H and they have exactly two common neighbors on H.*

Proof. Since a vertex with no neighbor in H, a cap or a pendant vertex is nested with any other vertex outside H, by Lemma 3.2, u and v are major or clones. If they are both major, there is a contradiction by Lemma 3.7. If u is a clone of y and v is major (or vice versa), then by Lemma 3.3, $vy \notin E(G)$, and it follows that u and v are nested, a contradiction. If u and v are both clones, then they have two or three common neighbors on H (because they cross). If they have three common neighbors, then G contains a square, a contradiction. Hence, they have two common neighbors as claimed. $\square$

Lemma 3.9 *Let H be a hole in a graph $G \in \mathcal{C}$ and let $P = u \dots v$ be a path of length at least 1, vertex-disjoint from H, and such that u and v have neighbors in H and no internal vertex of P has neighbor in H. If u and v are nested, then one of the following holds (up to a swap of u and v):*

(i) P has length 1, u is major or is a clone, and $N_H(v)$ is an edge that contains exactly one neighbor of u.

(ii) u is a major vertex or a clone, v is a cap and $N_H(v) \subseteq N_H(u)$.

(iii) $|N_H(v)| = 1$ and $N_H(v) \subseteq N_H(u)$.

(iv) $N_H(u) \cup N_H(v)$ is an edge of H.

Proof. By Lemma 3.2, u and v are major, clone, cap or pending. We may therefore consider four cases.

Case 1. At least one of u and v is major.

Up to symmetry, we suppose that u is major.

Suppose that v is also major. We apply Lemma 3.7 to u and v. Since u and v are nested, P has length at least 2. Hence G contains a theta, a contradiction. So, we may assume that v is minor.

Suppose that v is a clone of some vertex $x \in V(H)$. Since u and v are nested, $ux \notin E(G)$. By Lemma 3.5, P has length at least 2. So, $N_H(v)$ is included in some u-sector Q of H and P and $Q \setminus x$ form a theta, a contradiction.

Suppose that v is a cap and $N_H(v) = xy$. If x and y are in the interior of some u-sector Q of H, then P and Q form a pyramid. Hence, there exists a u-sector $R = u' \dots u''$ such that w.l.o.g. $x = u'$ and $y \in V(R)$. If $y = u''$ then (ii) holds. If $y \neq u''$, then P has length 1, for otherwise P and R form a pyramid. Hence, (i) holds.

Suppose that v is pending. Then (iii) holds for otherwise $N_H(v)$ is in the interior of some u-sector of H that together with P forms a theta.

Case 2. None of u, v is major, and at least one of u, v is a clone.

Up to symmetry, suppose that u is a clone of x.

Suppose that v is a clone of y. Since u and v are nested, we have $x \neq y$ and $xy \notin E(G)$. By Lemma 3.5, P has length at least 2, so P and $H \setminus \{x, y\}$ form a theta from u to v, a contradiction.

Suppose that v is a cap, and let yz be the two neighbors of v. If $x \in \{y, z\}$, then (ii) holds, so suppose $x \notin \{y, z\}$. Hence P and $H \setminus x$ form a pyramid, unless (i) holds.

Suppose that v is pending. Then (iii) holds for otherwise $H \setminus x$ and P form a theta.

Case 3. None of u, v is major or a clone, and at least one of u, v is a cap.

Up to symmetry, suppose that u is a cap.

Suppose that v is also a cap. Then (iv) holds, for otherwise H and P form a prism or an even wheel.

Suppose that v is pending. Then (iii) holds for otherwise H and P form a pyramid.

Case 4. Both u, v are pending vertices.

Then (iii) or (iv) holds, for otherwise H and P form a theta.

□

Lemma 3.10 *Let G be a graph in $\mathcal{C}$, H a hole in G and w a major vertex w.r.t. H. Suppose that a, w', b, w'' are four distinct vertices of H that appear in this order along H and such that w', w'' are adjacent to w. Then every path P of $G \setminus w$ from a to b has an internal vertex adjacent to w.*

Proof. Consider a counter-example such that P is of minimum length. Note that P has length at least 2. Let H_a (resp. H_b) be the path of H from w' to w'' that contains a (resp. b). Let $H_{w'}$ (resp. $H_{w''}$) be the path of H from a to b that contains w' (resp. w'').

(1) P^* is vertex-disjoint from $V(H) \cup \{w\}$.

Since P is a counterexample, its interior contains no neighbor of w, and since a, b, w' and w'' are distinct, we have $V(P) \cap \{w', w''\} = \emptyset$. So, an internal vertex of P that is in H would yield a smaller counterexample, a contradiction to the minimality of P. This proves (1).

We set $Q = P^* = u \dots v$, where u is adjacent to a and v is adjacent to b (possibly, $u = v$).

(2) u (resp. v) and w are nested.

Since P is a counterexample, u and w are non-adjacent. Since w is major, by Lemma 3.8, u and w are nested. Similarly, v and w are nested. This proves (2).

(3) u and v are distinct and nested w.r.t. H.

By (2), $N_H(u) \subseteq V(H_a)$ and $N_H(v) \subseteq V(H_b)$. So, u and v are distinct (because $a \notin N(v)$) and nested. This proves (3).

(4) We may assume that $H' = auQvbH_{w''}a$ is a hole that contains all neighbors of w in H except w'.

By the minimality of P, no internal vertex of Q has a neighbor in H_a^* or in H_b^*. It follows that $N_H(Q^*) \subseteq \{w', w''\}$.

Suppose first that $N_H(Q^*) = \{w', w''\}$. Then, H together with a path from w' to w'' with interior in Q^* form a theta from w' to w'', a contradiction.

Suppose now that $N_H(Q^*) = \emptyset$. Then, by (3), we may apply Lemma 3.9 to Q. Since $\{a\} \subseteq N_H(u) \subseteq V(H_a)$ and $\{b\} \subseteq N_H(v) \subseteq V(H_b)$, (ii), (iii) and (iv) of Lemma 3.9 cannot hold. So (i) of Lemma 3.9 must hold. Up to symmetry, we may therefore assume that w' is the unique common neighbor of u and v on H and $\{b\} = N_H(v) \setminus \{w'\}$. Since by (2) u and w are nested, w' is the unique neighbor of w in $H_{w'}$. Also, a may be chosen as close a possible to w'' along H_a, so that $H' = auQvbH_{w''}a$ is a hole that contains all neighbors of w in H except w'.

Suppose finally that $|N_H(Q^*)| = 1$. Up to symmetry we may assume $N_H(Q^*) = \{w'\}$. If w has a neighbor z in $H^*_{w'} \setminus w'$, then suppose up to symmetry that it is in $aH_{w'}w'$. We see that the four vertices a, z, w' and w'' appear in this order along H, so that a path from a to w' with interior in Q contradicts the minimality of P. It follows that w has no neighbor in $H^*_{w'} \setminus w'$. We may choose a and b closest to w'' along H_a and H_b respectively. Since by (2) u and w are nested (and v and w are nested), this implies that $H' = auQvbH_{w''}a$ is a hole that contains all neighbors of w in H except w'. This proves (4).

If w has exactly three neighbors in H, then by Lemma 3.1 they are pairwise non-adjacent and H' (from (4)) and w form a theta, a contradiction. So, w has at least five neighbors in H, so that (H', w) is a wheel. But then, one of (H, w) or (H', w) is an even wheel, a contradiction. □

We can now prove Theorem 2.4 restated below.

Theorem 2.4 *Let G be a graph in $\mathcal{C}$, H a hole in G and w a major vertex w.r.t. H. If C is a connected component of $G \setminus N[w]$, then there exists a w-sector $P = x \dots y$ of H such that $N(C) \subseteq \{x, y\} \cup (N(w) \setminus V(H))$.*

Proof. Set $W = N[w] \cap V(H)$ and $Z = N[w] \setminus V(H)$. Clearly, $N(C) \subseteq W \cup Z$. We have to prove that there exists a w-sector $P = x \dots y$ of H such that $N_W(C) \subseteq \{x, y\}$. Otherwise, we are in one of the following cases.
Case 1: there exists a, w', b, w'' in W, appearing in this order along H, with $a, b \in N_W(C)$. In this case, a path from a to b with interior in C contradicts Lemma 3.10.
Case 2: $|W| = 3$ and $N_W(C) = W = \{x, y, z\}$ (and by Lemma 3.1, x y and z are pairwise non-adjacent). In this case, suppose first that C contains a vertex a in $H \setminus W$. Up to symmetry, we may assume that a is in the w-sector of H from x to y. But then, x, a, y and z contradict Lemma 3.10 because C contains the interior of a path from a to z. Hence, $C \cap V(H) = \emptyset$. If some vertex v of C has more than one neighbor in $\{x, y, z\}$, then w and v are contained in a square of G, a contradiction. So, every vertex of C has at most one neighbor in W. Consider a path P with interior C and that is either from x to y, from y to z, or from z to x. Suppose that P has minimum length among all such paths. Up to symmetry, $P = x \dots y$, and by minimality, P contains no neighbor of z. It follows that P and H from a theta from x to y. □

4 Proper separators

A separator in a graph is *proper* if it is minimal and not a clique. In view of Theorem 2.3, we may restrict our attention to proper separators because it is known that in any graph G there exists at most $O(|V(G)|)$ minimal clique separators and that they can be enumrated in time $O(|V(G)||E(G)|)$, see [2] for details.

Our goal is to prove that a graph in $\mathcal{C}$ contains a "small" number of proper separators. This goal is achieved in the next section. Here we prove a series of technical lemmas telling where precisely the vertices of a proper separator are.

If C is a separator of G, a connected component D of $G \setminus C$ is *full* if every vertex of C has a neighbor in D.

Lemma 4.1 *If C is a proper separator of a graph $G \in \mathcal{C}$, then $G \setminus C$ has exactly two full connected components.*

Proof. Otherwise, let c_1c_2 be a non-edge in C and X, Y, Z be full components of $G \setminus C$. There exits a theta from c_1 to c_2, made of three paths with interior in X, Y and Z respectively. This is a contradiction. $\square$

In what follows, when C is a proper separator, we denote by L and R the two full components of $G \setminus C$ that exist by Lemma 4.1. We call a C-hole any hole H such that $V(H) \cap C = \{c_1, c_2\}$ where c_1, c_2 are non-adjacent vertices, one component of $H \setminus \{c_1, c_2\}$ is in L and the other one is in R. We then say that H is a (C, c_1, c_2)-hole.

For a (C, c_1, c_2)-hole H, we use notation H_L for the path of H from c_1 to c_2 with interior in L and H_R for the path of H from c_1 to c_2 with interior in R. We let l_1 be the neighbor of c_1 in H_L. We define similarly vertices r_1, l_2, and r_2.

A C-hole H is *clean* w.r.t. C if every major vertex w.r.t. H is in C. The next lemma shows that clean holes exist.

Lemma 4.2 *Let C be a proper separator of a graph $G \in \mathcal{C}$. If c_1 and c_2 are non-adjacent vertices of C, then a shortest (C, c_1, c_2)-hole H is clean w.r.t. C*

Proof. Consider a vertex $v \notin C$ that is major w.r.t H. If $N_H(v) \subseteq V(H_L)$, then a shorter (C, c_1, c_2)-hole exists (using v as a shortcut), a contradiction. Similarly, there is a contradiction if $N_H(v) \subseteq V(H_R)$. It follows that v has neighbors in both H_L^* and H_R^*, and in particular in both L and R, a contradiction. This proves that H is clean w.r.t. C. $\square$

Let C be a proper separator of a graph G and H be a C-hole. A vertex in G is (C, H)-*heavy* if it is major w.r.t. H and has neighbors in the interiors of both H_L and H_R. Observe that a (C, H)-heavy vertex must be in C, because it has neighbors in both L and R.

Lemma 4.3 *Let C be a proper separator of a graph $G \in \mathcal{C}$. Let H and H' be two (C, c_1, c_2)-holes that are clean w.r.t. C. A vertex in C is (C, H)-heavy if and only if it is (C, H')-heavy.*

Proof. Otherwise, suppose up to symmetry that some vertex v is (C, H)-heavy and not (C, H')-heavy. Hence, v has a neighbor v_L in the interior of H_L and a neighbor v_R in the interior of H_R.

(1) We may assume that v is a clone of c_1 w.r.t. H'.

The vertices c_1, v_L, c_2, v_R are distinct and appear in this order along H. By Lemma 3.10, the path H'_L has an internal vertex adjacent to v. Similarly, H'_R has an internal vertex adjacent to v. Since v is not (C, H')-heavy, the only possibility is that v is a clone of c_1 or c_2 w.r.t. H', and up to symmetry, we suppose it is a clone of c_1. This proves (1).

(2) We may assume that v_L is an internal vertex of $l_1H_Lc_2$ (in particular, H_L has length at least 3).

Since v is not a clone w.r.t. H, $N_H(v) \not\subseteq \{c_1, l_1, r_1\}$. Since by (1) $vc_2 \notin E(G)$, v has a neighbor in the interior of either $l_1H_Lc_2$ or $r_1H_Rc_2$. Up to symmetry, we may assume that v has a neighbor in the interior of $l_1H_Lc_2$. Hence, v_L can be chosen in the interior of $l_1H_Lc_2$. This proves (2).

(3) $l_1 \neq l'_1$.

Otherwise the vertices l_1, v_L, c_2, v_R are distinct and appear in this order along H. By Lemma 3.10, the path $l_1H'_Lc_2$ has an internal vertex adjacent to v, a contradiction to (1). This proves (3).

(4) $\{c_1\} \subseteq N_H(l'_1) \subseteq \{c_1, l_1\}$.

Otherwise l'_1 has two non-adjacent neighbors in H_L, and since H is clean w.r.t. C, by Lemma 3.2, l'_1 is a clone of l_1 w.r.t. H. Hence, the hole $H_{l'_1 \setminus l_1}$ contains four distinct vertices (namely l'_1, v_L, c_2, v_R). By Lemma 3.10, v has a neighbor in the interior of $l'_1H'_Lc_2$. This contradicts (1). This proves (4).

By (1), v is not adjacent to c_2. It follows that c_2 is an internal vertex of some v-sector Q of H. We set $Q = q_L \dots q_R$ with $q_L \in L$ and $q_R \in R$. Note that by (2), $q_L \neq l_1$. Let x be the vertex of H'_L with a neighbor in Q, closest to c_1 along H'_L. Note that x exists because of c_2. We set $S = l'_1H'_Lx$.

(5) S has length at least 1.

Otherwise S has length zero, so $x = l'_1$ and l'_1 has a neighbor in Q. This contradicts (4). This proves (5).

(6) S is vertex disjoint from H and the only edges between S and H are l'_1c_1, possibly l'_1l_1, and the edges between x and Q.

By (3), $l'_1 \notin V(H)$ and by (4), the only edges between l'_1 and Q are l'_1c_1 and possibly l'_1l_1. Note that by the definition of x, S is vertex disjoint from Q and x is the only vertex of S with neighbors in Q. Suppose that S contains any vertex b of H or that there is any edge ab with $a \in V(S)$, $b \in V(H)$ and ab is not l'_1c_1, l'_1l_1 or an edge between x and Q. Then consider the four distinct vertices of H: c_2, q_L, b and v_R. We see that $V(H'_L) \cup \{b\}$ contains a path P from c_2 to b. By (1), P contains no internal vertex adjacent to v. This contradicts Lemma 3.10. This proves (6).

By (5) and (6), S and H contradict Lemma 3.9. □

By Lemma 4.3, for a vertex not in C, being heavy does not depend on the choice of a particular hole, but only on the choice of C, c_1 and c_2. The notion of (C, c_1, c_2)-heavy vertex is therefore relevant: a vertex is (C, c_1, c_2)-*heavy* if for some (or equivalently every) clean (C, c_1, c_2)-hole H, it is (C, H)-heavy.

Until the end of the section, we do not recall in the statements of the lemmas that we deal with a graph G in $\mathcal{C}$, a proper separator C, a clean (C, c_1, c_2)-hole H with the following notation: l_1 is the neighbor of c_1 in H_L and l'_1 is the neighbor of l_1 in $H_L \setminus c_1$. We define similarly vertices r_1, r'_1, l_2, l'_2, r_2 and r'_2.

For $i = 1, 2$, we denote by L_i the set made of l_i and all the clones of l_i w.r.t. H. We denote by R_i the set made of r_i and all the clones of r_i w.r.t. H. We denote by C_i the set of vertices of G that are not (C, c_1, c_2)-heavy and have neighbors in both L_i and R_i (observe that $c_i \in C_i$). Note that possibly $L_1 = L_2$ or $R_1 = R_2$ (not both since H is not a C_4). Observe that L_i is possibly not included in L, because some vertices of L_i can be in C. Similarly, R_i is possibly not included in R. And C_i is possibly not included in C because some vertices of C_i might be in L or in R (not in both, because as we will see, C_i is clique and L is anticomplete to R).

Lemma 4.4 *For $i \in \{1, 2\}$, L_i, R_i and C_i are pairwise disjoint cliques. Moreover, L_i is anticomplete to R_i, and C_i is anticomplete to $H \setminus \{c_i, l_i, r_i\}$.*

Proof. We prove the lemma for $i = 1$ ($i = 2$ is similar). Clearly, L_1 and R_1 are disjoint. They are cliques for otherwise, G contains a square. By Lemma 3.5, L_1 is anticomplete to R_1. It follows that C_1 is disjoint from both L_1 and R_1.

Let us prove that C_1 is anticomplete to $H \setminus \{c_1, l_1, r_1\}$. Otherwise, let $c \in C_1$ be a vertex with some neighbor in $H \setminus \{c_1, l_1, r_1\}$. Note that $c \neq c_1$. Also, $cc_1 \in E(G)$ for otherwise G contains a square (with c, c_1, and neighbors of c in L_1 and R_1). Let $l \in L_1$ and $r \in R_1$ be neighbors of c (they exists by definition of C_1). We see that c is major w.r.t. $H' = (H_{l/l_1})_{r/r_1}$. Hence, by Lemma 3.3 (applied twice) c is major w.r.t. H, and therefore (C, c_1, c_2)-heavy, a contradiction to the definition of C_1.

It remains to prove that C_1 is a clique, so suppose for a contradiction that c and c' are non-adjacent vertices of C_1. Let l, r be neighbors of c in L_1, R_1 respectively, and l', r' be neighbors of c' in L_1, R_1 respectively. If c and c' have common neighbors in both L_1 and R_1, then G contains a square, a contradiction. Hence, we may assume that $l \neq l'$ and that $cl', c'l \notin E(G)$.

If c and c' have a common neighbor $r'' \in R_1$, then the paths $r''cl$, $r''c'l'$ and $r''r'_1Pl'_1$

form a pyramid. So, $r \neq r'$, cr' and $c'r \notin E(G)$. Hence, the paths rcl, $rc'l'$ and P form a prism. □

Lemma 4.5 *For $i \in \{1,2\}$, L_i, R_i and C_i are pairwise disjoint cliques. Moreover, L_i is anticomplete to R_i, and C_i is anticomplete to $H \setminus \{c_i, l_i, r_i\}$.*

Proof. We prove the lemma for $i = 1$ ($i = 2$ is similar). Clearly, L_1 and R_1 are disjoint. They are cliques for otherwise, G contains a square. By Lemma 3.5, L_1 is anticomplete to R_1. It follows that C_1 is disjoint from both L_1 and R_1.

Let us prove that C_1 is anticomplete to $H \setminus \{c_1, l_1, r_1\}$. Otherwise, let $c \in C_1$ be a vertex with some neighbor in $H \setminus \{c_1, l_1, r_1\}$. Note that $c \neq c_1$. Also, $cc_1 \in E(G)$ for otherwise G contains a square (with c, c_1, and neighbors of c in L_1 and R_1). Let $l \in L_1$ be a neighbor of c (it exists by definition of C_1). We set $P = r'_1 H_R c_2 H_L l'_1$ (it has length at least 1, and is induced by $V(H) \setminus \{l_1, c_1, r_1\}$). So, c has a neighbor in P and we let x be the neighbor of c in P closest to l'_1 along P. We see that if $cl_1 \notin E(G)$, then $l \neq l_1$ and the hole $cxPl'_1l_1c_1c$ is the rim of an even wheel with center l, a contradiction. Hence, $cl_1 \in E(G)$, and symmetrically $cr_1 \in E(G)$. It follows that c is major w.r.t. H and has neighbors in both H^*_L and H^*_R, a contradiction to the definition of C_1.

It remains to prove that C_1 is a clique, so suppose for a contradiction that c and c' are non-adjacent vertices of C_1. Let l, r be neighbors of c in L_1, R_1 respectively, and l', r' be neighbors of c' in L_1, R_1 respectively. If c and c' have common neighbors in both L_1 and R_1, then G contains a square, a contradiction. Hence, we may assume that $l \neq l'$ and that $cl', c'l \notin E(G)$.

If c and c' have a common neighbor $r'' \in R_1$, then the paths $r''cl$, $r''c'l'$ and $r''r'_1Pl'_1$ form a pyramid. So, $r \neq r'$, cr' and $c'r \notin E(G)$. Hence, the paths rcl, $rc'l'$ and P form a prism. □

For $i \in \{1,2\}$, an (L,i)*-viaduct w.r.t.* (C,H) is a path $Q = u_L \dots u_R$ of G such that:

(i) $V(Q^*) \cap (V(H) \cup L_i \cup R_i) = \emptyset$;

(ii) $V(Q^*)$ is anticomplete to $V(H \setminus c_i)$;

(iii) $C \cap V(Q) = \{u_L\}$;

(iv) $u_R \in R_i \setminus C$ (so possibly, $u_R = r_i$);

(v) one of the following holds:

- u_L is major w.r.t. H, $N_H(u_L) \subseteq V(H_L)$, $u_Lc_i \in E(G)$; or
- $u_L \in L_i \cap C$.

For $i \in \{1,2\}$, an (R,i)*-viaduct w.r.t.* (C,H) is a path $Q = u_L \dots u_R$ of G such that:

(i) $V(Q^*) \cap (V(H) \cup L_i \cup R_i) = \emptyset$;

(ii) $V(Q^*)$ is anticomplete to $V(H \setminus c_i)$;

(iii) $C \cap V(Q) = \{u_R\}$;

(iv) $u_L \in L_i \setminus C$ (so possibly, $u_L = l_i$);

(v) one of the following holds:

- u_R is major w.r.t. H, $N_H(u_R) \subseteq V(H_R)$, $u_R c_i \in E(G)$; or
- $u_R \in R_i \cap C$.

We call *viaduct* any path that is an (L, i)-viaduct or an (R, i)-viaduct for $i \in \{1, 2\}$.

Lemma 4.6 *For every $i \in \{1, 2\}$, every (L, i)-viaduct and (R, i)-viaduct has length at least 2 and contains an odd number of neighbors of c_i (at least 3).*

Proof. Suppose $Q = u_L \dots u_R$ is an $(L, 1)$-viaduct (the proof is similar for other types of viaducts).

Since $u_L \in C$ and $u_R \notin C$, we have $u_L \neq u_R$. So Q has length at least 1, and suppose for a contradiction that is has length 1. Then clearly $u_R \neq r_1$ and since u_L and u_R are nested, by Lemma 3.9 applied to Q and H, and (i) is the only possible outcome. So, one end of Q is a cap, a contradiction to the definition of viaducts (observe however that if u, v are vertices like in outcome (i) Lemma 3.9, then uvr_1 is possibly a viaduct of length 2). So Q has length at least 2.

Observe that $u_L \notin V(H)$ (while u_R is either in $V(H)$ or is a clone of r_1). Let x be the neighbor of u_L in H_L, closest to c_2 along H_L. Consider the hole J induced by $V(Q) \cup V(xH_Lc_2) \cup (V(H_R) \setminus \{c_1, r_1\})$ (note that r_1 may be in J, when $r_1 = u_R$). Now, $c_1 \notin V(J)$ and c_1 contains two non-adjacent neighbors in J (namely u_L and u_R), hence, by Lemmas 3.1 and 3.2, c is a clone or a major vertex w.r.t. J, and it therefore has an odd number of neighbors in J (at least 3). □

The *potential* of (C, c_1, c_2) is the number of (C, c_1, c_2)-heavy vertices. The main result of this section is the following.

Lemma 4.7 *Let C be a proper separator of a graph $G \in \mathcal{C}$. Let c_1 and c_2 be non-adjacent vertices of C, chosen such that the potential of (C, c_1, c_2) is maximum. Let H be a clean (C, c_1, c_2)-hole. If $c \in C \setminus \{c_1, c_2\}$, then one of the following statements holds:*

(i) c is (C, H)-heavy;

(ii) For some $i \in \{1, 2\}$, c has a neighbor in $L_i \setminus C$ and a neighbor in $R_i \setminus C$;

(iii) c is the end of some viaduct w.r.t. (C, H).

Proof. Since C is a proper separator and L is connected, there exists a path $Q_L = c \dots c_L$ such that $V(Q_L \setminus c) \subseteq L$ and c_L has neighbors in the interior of H_L (possibly $c = c_L$). There exits a similar path $Q_R = c \dots c_R$. We set $Q = c_L Q_L c Q_R c_R$ and suppose that Q is minimal (so Q_L and Q_R are shortest paths).

(1) We may assume that Q has length at least 1. In particular, c_L and c_R are nested w.r.t. H.

Otherwise, $Q = c = c_L = c_R$. Since c has a neighbor in H_L^* and in H_R^*, it is either a major vertex or a clone. If it is major, then it is heavy w.r.t. H and (i) holds. If it is a clone, it must be a clone of c_1 or c_2, so (ii) holds. This proves (1).

(2) We may assume that c_1 has neighbors in the interior of Q and c_2 has no neighbors in the interior of Q.

Suppose that both c_1 and c_2 have neighbors in the interior of Q. Then, H and a shortest path from c_1 to c_2 with interior in the interior of Q form a theta, a contradiction.

So, suppose that none of c_1, c_2 have neighbors in the interior of Q. Since by (1) c_L and c_R are nested w.r.t. H, we apply Lemma 3.9 to Q. Since c_L has neighbors in the interior of H_L and c_R has neighbors in the interior of H_R, outcomes (ii), (iii) and (iv) cannot hold.

Hence outcome (i) holds. So $Q = c_L c_R$, $c \in \{c_L, c_R\}$ and exactly one of c_1 or c_2 (say c_1) is a common neighbor of c_L and c_R, and up to symmetry, c_L is major or clone of l_1, and c_R is a cap. If $c = c_L$, then $c_L c_R r_1$ is an $(L, 1)$-viaduct and (iii) holds. If $c = c_R$, then since H is clean, c_L cannot be major, so it is a clone, $c_L \in L_1 \setminus C$, $cr_1 \in E(G)$ so (ii) holds.

Hence, we may assume that exactly one of c_1 or c_2 has neighbors in the interior of Q, and up to symmetry, we may assume that it is c_1. This proves (2).

(3) If $c_L c_1 \notin E(G)$, then $N_H(c_L) = \{l_1\}$.

Let x_L be the neighbor of c_1 in Q, closest to c_L along Q (x_L exists by (2) and $x_L \neq c_L$ by assumption). Since c_L and x_L are nested (because x_L has no neighbor in the interior of H_L), we may apply Lemma 3.9 to $c_L Q x_L$. Since c_L and x_L have no common neighbor on H, (i), (ii) and (iii) do not hold. Hence (iv) holds and $N_H(c_L) = \{l_1\}$. This proves (3).

(4) If c_L is a cap or a pending vertex, then $\{l_1\} \subseteq N_H(c_L) \subseteq \{l_1, c_1\}$.
If c_R is a cap or a pending vertex, then $\{r_1\} \subseteq N_H(c_R) \subseteq \{r_1, c_1\}$.

If $c_L c_1 \notin E(G)$, then our claim holds by (3). Otherwise, $c_L c_1 \in E(G)$, c_L must be a cap and $N_H(c_L) = \{c_1, l_1\}$. The proof is similar for the claim about c_R. This proves (4).

(5) If c_L is a clone or a major vertex w.r.t. H, then $c_1 c_L \in E(G)$ and either $c_L \in C$ or $c_L \in L_1 \setminus C$.
The analogous statement holds for c_R.

By symmetry, it suffices to prove the statement for c_L, so assume that c_L is a clone or a major vertex w.r.t. H. By (3), $c_L c_1 \in E(G)$. If c_L is major then $c_L \in C$ because H is clean w.r.t. C. If c_L is clone and $c_L \notin C$, then $c_L \in L_1 \setminus C$ because c_L has no neighbors in R. This proves (5).

(6) We may assume that $c_L \in L_1 \setminus C$ or $\{l_1\} \subseteq N_H(c_L) \subseteq \{l_1, c_1\}$, and $c_R \in R_1 \setminus C$ or $\{r_1\} \subseteq N_H(c_R) \subseteq \{r_1, c_1\}$.

By symmetry it suffices to prove the statement about c_L. If c_L is a pending vertex or a cap, then the result follows by (4). So, suppose that c_L is a clone or a major vertex. By (5), $c_1c_L \in E(G)$ and either $c_L \in C$ or $c_L \in L_1 \setminus C$. We may assume that $c_L \in C$, and hence $c_R \notin C$. Note that by (1) and since $c_1c_L \in E(G)$, if c_L is a clone w.r.t. H, then it is a clone of l_1, and if it is major then $N_H(c_L) \subseteq V(H_L)$.

If c_R is pending or cap w.r.t. H, then by (4), $\{r_1\} \subseteq N(c_R) \cap V(H) \subseteq \{r_1, c_1\}$ and hence $c_LQc_Rr_1$ is an $(L,1)$-viaduct and (iii) holds. So, we may assume that c_R is a clone or a major vertex w.r.t. H. By (5) and since $c_R \notin C$, it follows that $c_R \in R_1$. But then Q is an $(L,1)$-viaduct and (iii) holds. This proves (6).

By (6), $\{l_1\} \subseteq N_H(c_L) \subseteq \{l'_1, l_1, c_1\}$ and $\{r_1\} \subseteq N_H(c_R) \subseteq \{r'_1, r_1, c_1\}$. So, $V(Q) \cup V(H) \setminus \{c_1\}$ contains a hole that contains Q and c_2, which we denote by J. Note that J is a (C, c, c_2)-hole.

(7) J is a clean w.r.t. C.

Otherwise, let $d \notin C$ be a vertex that is major w.r.t. J. By symmetry, we may assume that $d \in L$.

Since d is major w.r.t. J and not major w.r.t. H (since H is clean), d must have a neighbor in Q. Let d_L (resp. d_R) be the neighbor of d in Q that is closest to c_L (resp. c_R) along Q. Note that $d_L, d_R \in V(c_LQc)$ since $d \in L$. If d_LQd_R is of length greater than 2, then $V(Q_L) \cup \{d\}$ contains a path from c to c_L that is shorter than Q_L, contradicting the minimality of Q_L. So, d_LQd_R is of length at most 2.

Since d is major w.r.t. J, it follows that $N_J(d) \not\subseteq V(Q)$. Suppose that $N_J(d) \subseteq V(Q) \cup \{c_2\}$. So d is adjacent to c_2. By Lemma 3.1, it follows that d has exactly three neighbors in J that are furthermore pairwise non-adjacent, namely d_L, d_R and c_2. But then d_LQd_R and d form a square. Therefore d has a neighbor in $H_L \setminus \{c_1, c_2\}$. By minimality of Q_L, it follows that $N_Q(d) \subseteq \{c_L, c'_L\}$, where c'_L is the neighbor of c_L in Q. By Lemma 3.1 applied to d and J, d has two non-adjacent neighbors in $J \setminus Q$. Since $V(J \setminus Q) \subseteq V(H)$ and d is not major w.r.t. H, it follows that d is a clone of some vertex d' w.r.t. H, where d' is an internal vertex of $J \setminus Q$ (so $d' \notin \{c_1, c_2, l_1\}$ and $d' = l'_1$ is possibly only when c_L is not a clone). So, by Lemma 3.1, $N_Q(d) = \{c_L, c'_L\}$. If $c_L \in L_1$, then $(H_{c_L \setminus l_1}, d)$ is an even wheel. So $N_H(c_L) \subseteq \{l_1, c_1\}$. Note that by minimality of Q, no internal vertex of Q has a neighbor in $H \setminus \{c_1, c_2\}$. Let c'_1 be the neighbor of c_1 in the interior of Q that is closest to c_L along Q (it exists by (2)). If $N_H(c_L) = \{l_1\}$, then $H_{d \setminus d'}$ and $c'_LQc'_1$ form a theta from d to c_1. So $N_H(c_L) = \{l_1, c_1\}$. Let D be the path from d to l_1 contained in $(H \setminus \{d', c_2\}) \cup \{d\}$. Then D and $c_LQc'_1$ form an even wheel with center c_L. This proves (7).

(8) Let $d \in C \setminus \{c, c_1, c_2\}$. If d is (C, H)-heavy, then d is (C, J)-heavy.

For suppose that d is (C, H)-heavy but not (C, J)-heavy. So d is major w.r.t. H and has neighbors in both H_L^* and H_R^*.

Suppose that d does not have a neighbor in J_L^*. Then $c_L \in L_1 \setminus C$, $\{l_1\} \subseteq N_{H_L}(d) \subseteq \{l_1, c_1\}$, and d is not adjacent to c_L. But then since d is major w.r.t. H, by Lemma 3.1, it follows that $H_{c_L \setminus l_1}$ and d form either an even wheel with center d or a theta. So d has a neighbor in J_L^*, and by symmetry d has a neighbor in J_R^*. Since d is not major w.r.t. J, it follows that d is a clone of c or c_2 w.r.t. J.

If d is a clone of c w.r.t. J, then $N_H(d) \subseteq \{l_1, c_1, r_1\}$ contradicting the assumption that d is major w.r.t. H. So d is a clone of c_2 w.r.t. J.

Since H is of length greater than 4, w.l.o.g. H_L is of length greater than 2. In particular, J contains l_2. Since d is major w.r.t. H and H contains l_2 and c_2, by Lemma 3.1, d has at least five neighbors in H. It follows that d is adacent to l_1 or r_1. If d is adjacent to l_1, then $c_L \in L_1 \setminus C$ and d is not adjacent to c_L. But then $(H_{c_L \setminus l_1}, d)$ is an even wheel. So d is not adjacent to l_1 and hence $N_H(d) = \{l_2, c_2, r_2, r_1, c_1\}$. In particular, $r_1 \neq r_2$, i.e. H_R is of length greater than 2. But then, we get a contradiction by a symmetric argument. This proves (8).

To conclude the proof, by (7) J is clean w.r.t. C. Also, c_1 has neighbor in J_L^* and J_R^*, and is therefore major w.r.t. J or a clone of c. In this last case, $c_L \in L_1 \setminus C$, $c_R \in R_1 \setminus C$ and Q has length 2, so (ii) holds. Hence, we may assume that c_1 is major w.r.t. J.

Note that c is not major w.r.t. H. By (8), we see that the number of (C, J)-heavy vertices is greater than the number of (C, H)-heavy vertices. So, by Lemma 4.3, the potential of (C, c, c_2) is greater than then potential of (C, c_1, c_2), a contradiction to the choice of c_1 and c_2. □

Lemma 4.8 *For $i \in \{1, 2\}$, there does not exist both an (L, i)-viaduct and an (R, i)-viaduct. In particular, at least one of L_i, R_i contains no vertex of C.*

Proof. Suppose there exists an $(L, 1)$-viaduct $P = u_L \dots u_R$ and an $(R, 1)$-viaduct $Q = v_L \dots v_R$ (the case where $i = 2$ is similar). Then, $u_L \in C$, $u_R \in R_1$, $v_L \in L_1$ and $v_R \in C$. Note that $N_H(u_L) \subseteq V(H_L)$ and $N_H(v_R) \subseteq V(H_R)$.

(1) $V(P \setminus u_L)$ and $V(Q \setminus v_R)$ are disjoint and anticomplete. Moreover, $u_L v_R \notin E(G)$.

The first claim is because L and R are connected components of $G \setminus C$ and $V(P \setminus u_L) \subseteq R$ and $V(Q \setminus v_R) \subseteq L$.

Since u_L and v_R are nested and both major or clones w.r.t. H, $u_L v_R \notin E(G)$ follows from Lemma 3.5. This proves (1).

Let x_L be the neighbor of u_L in $v_L l_1' H_L c_2$, closest to v_L along this path. Let y_L be the neighbor of u_L in $v_L l_1' H_L c_2$, closest to c_2 along this path. Note that x_L and y_L exist and are distinct from the definition of viaducts and Lemma 4.6. (but possibly, $x_L = v_L$,

$y_L = l'_1$ and $x_L y_L \in E(G)$ when u_L is a clone w.r.t. H). Let x_R be the neighbor of v_R in $u_R r'_1 H_R c_2$, closest to u_R along this path. Let y_R be the neighbor of v_R in $u_R r'_1 H_R c_2$, closest to c_2 along this path.

If $x_L \neq v_L$, we set $S_L = u_L x_L H_L l'_1 v_L$. If $x_L = v_L$ we set $S_L = u_L v_L$. If $x_R \neq u_R$, we set $S_R = v_R x_R H_L r'_1 u_R$. If $x_R = u_R$ we set $S_R = v_R u_R$.

By (1), $J = u_L S_L v_L Q v_R S_R u_R P u_L$ is a cycle whose only possible chords are edges from u_L to $Q \setminus v_L$ and from v_R to $P \setminus u_R$. And such chords exist for otherwise, J is a hole and by Lemma 4.6, c_1 has an even number of neighbors in J.

Up to the symmetry between P and Q, we suppose that v_R has a neighbor in $P \setminus u_R$ and let p be the neighbor of v_R in P closest to u_L along P (note that by (1), $p \neq u_L$).

If u_L has a neighbor in $Q \setminus v_L$, then let q be the neighbor of u_L in Q closest to v_R along Q (note that by (1), $q \neq v_R$). We see that the three paths $v_R p P u_L$, $v_R Q q u_L$ and $v_R y_R H_R c_2 H_L y_L u_L$ form theta, a contradiction. Hence, u_L has no neighbor in $Q \setminus v_L$.

If $x_L y_L \notin E(G)$, then the three paths $v_R p P u_L$, $v_R Q v_L S_L x_L u_L$ and $v_R y_R H_R c_2 H_L y_L u_L$ form a theta. If $x_L y_L \in E(G)$, then $x_L = v_L$ and $y_L = l'_1$, so the three paths $v_R p P u_L$, $v_R Q v_L$ and $v_R y_R H_R c_2 H_L l'_1$ form a pyramid. In every case, there is a contradiction. □

Lemma 4.9 *Let c_1, c_2, and H be as in Lemma 4.7. Let $i \in \{1, 2\}$.*

- *Suppose $R_i \cap C = \emptyset$. Then there exists a vertex $c \in C_i$ such that for all $x \in C_i$, $x \in C$ if and only if $N_{L_i}(c) \subseteq N_{L_i}(x)$.*
- *Suppose $L_i \cap C = \emptyset$. Then there exists a vertex $c \in C_i$ such that for all $x \in C_i$, $x \in C$ if and only if $N_{R_i}(c) \subseteq N_{R_i}(x)$.*

Proof. By symmetry, it is enough to prove the first claim for $i = 1$, so suppose $R_1 \cap C = \emptyset$. Note that $C_1 \cap C \neq \emptyset$ since $c_1 \in C_1$. Let $c \in C_1 \cap C$ be such that $N_{L_1}(c)$ is minimal (inclusion wise). By Lemma 4.5, $N_H(c) \subseteq \{c_1, l_1, r_1\}$, and hence by Lemma 4.7 applied to c, c has a neighbor l in $L_1 \setminus C$ and a neighbor r in $R_1 \setminus C$. Let $x \in C_1$.

If $N_{L_1}(c) \subseteq N_{L_1}(x)$, then x is adjacent to $l \in L_1 \setminus C \subseteq L$ and to some vertex of $R_1 = R_1 \setminus C \subseteq R$. Hence, $x \in C$.

Conversely, suppose that $x \notin C$ and $N_{L_1}(c) \not\subseteq N_{L_1}(x)$. This means that c has a neighbor y in L_1 that is not adjacent to x. If x has a neighbor z in $L_1 \setminus N(c)$, then $xzycx$ is a square by Lemma 4.5, a contradiction. Hence, $N_{L_1}(x) \subsetneq N_{L_1}(c)$, contradicting the choice of c. □

5 The main proof

We describe two algorithms $\mathcal{A}_{L,L}$ and $\mathcal{A}_{L,R}$ that enumerate some proper separators of an input graph G. Note that these algorithms can be applied to any graph. Algorithm $\mathcal{A}_{L,L}$

is described in Table 1. Algorithm $\mathcal{A}_{L,R}$ is very similar to $\mathcal{A}_{L,L}$, only steps 9.–14. slightly differ (the roles of L_2 and R_2 are swapped). In Table 2 we indicate what are these steps.

Lemma 5.1 *Let G be a graph in $\mathcal{C}$. If we run the two algorithms $\mathcal{A}_{L,L}$ and $\mathcal{A}_{L,R}$ on G, then the output is the list of all proper separators of G and the running time is at most $O(|V(G)|^{10})$.*

Proof. Because of step 14., the algorithm obviously outputs a list of proper separators of G. Conversely, consider a proper separator D, and let us check that at least one of $\mathcal{A}_{L,L}$ or $\mathcal{A}_{L,R}$ outputs D.

Let c_1, c_2 be vertices in D such that the potential of (D, c_1, c_2) is maximum. At some point, in step 1., the algorithm considers the pair of vertices (c_1, c_2) and correctly puts c_1 and c_2 in C. At this step, $C \subseteq D$.

Let J be a shortest (D, c_1, c_2)-hole. Note that by Lemma 4.2, J is clean. We use the notation l_1, r_1, l_2, r_2 as in section 4. At some point in step 2., the algorithm considers the 4-tuple (l_1, r_1, l_2, r_2).

In step 3., we claim that all (D, c_1, c_2)-heavy vertices are put in C. Indeed, let v be such a (D, c_1, c_2)-heavy vertex. Note that by Lemma 4.3, v is (D, J)-heavy and hence c_1 and c_2 do not belong to the same v-sector of J. By Theorem 2.4 applied to J and v, vertices c_1 and c_2 are in different components of $G \setminus (N[v] \setminus \{c_1, c_2\})$. Note that all (D, J)-heavy vertices are obviously in D, since they have neighbors in both L and R. Hence, at this step, we have $C \subseteq D$. Note that if l_1 say is put in C at this step, then l_1 is adjacent to some vertex of R, a contradiction, so it is correct to discard C. There is a similar argument for r_1, l_2, r_2.

We claim that in step 4. a clean (D, c_1, c_2)-hole is computed. Indeed, by Lemma 4.7 applied to J, all vertices of D are either heavy (and these are already in C), or adjacent to c_1 or c_2 from definitions of viaducts and by Lemma 4.5. So, when we compute the paths, all vertices of D are removed. Note that the paths exist, because of J. Hence, $V(H^*_L) \subseteq L$ and $V(H^*_R) \subseteq R$ showing that H is a hole. In particular, it is correct to discard C when H is not a hole. Clearly, H is a (D, c_1, c_2)-hole. Since H and J both go through c_1, l_1, r_1, c_2, l_2, r_2 and J is a shortest (D, c_1, c_2)-hole, the length of H is the same as the length of J, and hence H is also a shortest (D, c_1, c_2)-hole. So, by Lemma 4.2, H is clean w.r.t. D. Note that H and J are potentially different holes, but they both go through the same vertices $c_1, l_1, r_1, c_2, l_2, r_2$ and they have the same heavy vertices by Lemma 4.3.

Since H is clean, in step 5. it is correct to put in C every vertex that is major w.r.t. H. We still have $C \subseteq D$.

In step 8., we know by Lemma 4.5 (and since H is not a square) that discarding C is correct whenever we have to do so.

Now, by Lemma 4.8 and the symmetry between L and R we may assume that $R_1 \cap D = \emptyset$. More precisely, if $R_1 \cap D \neq \emptyset$ then $L_1 \cap D = \emptyset$, so at some other step of enumeration, the 4-tuple (r_1, l_1, r_2, l_2) (and not (l_1, r_1, l_2, r_2)) is considered, so that $R_1 \cap D = \emptyset$.

1. Enumerate all pairs of distinct and non-adjacent vertices (c_1, c_2) of G.
   Set $C = \{c_1, c_2\}$.
2. Enumerate all 4-tuple of vertices (l_1, r_1, l_2, r_2) such that $\{l_1, c_1, r_1\}$ and $\{l_2, c_2, r_2\}$ both induce a path of length 2 and $\{l_1, l_2\}$ is anticomplete to $\{r_1, r_2\}$. Note that possibly $l_1 = l_2$ or $r_1 = r_2$.
3. Add to C every vertex v such that c_1 and c_2 are in two distinct connected components of $G \setminus (N[v] \setminus \{c_1, c_2\})$. If $\{l_1, r_1, l_2, r_2\} \cap C \neq \emptyset$, discard C.
4. In $G \setminus ((C \cup N(c_1) \cup N(c_2)) \setminus \{l_1, r_1, l_2, r_2\})$, compute a shortest path H_L from l_1 to l_2 and a shortest path H_R from r_1 to r_2. Set $H = c_1 r_1 H_R r_2 c_2 l_2 H_L l_1 c_1$. If H is not a hole, discard C.
5. Add to C every vertex that is major w.r.t. H.
6. Compute the set L_1 of clones of l_1 w.r.t. H and add l_1 to L_1. Compute similar sets R_1, L_2 and R_2.
7. Compute the set C_1 of vertices that have neighbors in both L_1 and R_1 and that are not major w.r.t. H. Note that $c_1 \in C_1$. Compute a similar set C_2.
8. Check that $L_1, C_1, R_1, L_2, C_2, R_2$ are disjoint cliques, except that possibly exactly one of the equalities $L_1 = L_2$ and $R_1 = R_2$ holds. If this check fails, discard C.
9. Enumerate all pairs of vertices $c_1' \in C_1$, $c_2' \in C_2$.
10. Add to C all vertices x from C_1 such that $N_{L_1}(c_1') \subseteq N_{L_1}(x)$.
11. Add to C all vertices x from C_2 such that $N_{L_2}(c_2') \subseteq N_{L_2}(x)$.
12. Add to C every vertex $c \in L_1$ such that there exists a path $Q = c \dots v$ with the following properties: $v \in R_1$, $V(Q) \cap C = \emptyset$, $V(Q^*) \cap (V(H) \cup L_i \cup R_i) = \emptyset$ and Q^* is anticomplete to $H \setminus c_1$.
13. Add to C every vertex $c \in L_2$ such that there exists a path $Q = c \dots v$ with the following properties: $v \in R_2$, $V(Q) \cap C = \emptyset$, $V(Q^*) \cap (V(H) \cup L_2 \cup R_2) = \emptyset$ and Q^* is anticomplete to $H \setminus c_2$.
14. Check whether C is proper separator of G. If not, discard C. Otherwise return C, and go to the next step of enumeration (so step 1., 2. or 9.).

Table 1: Algorithm $\mathcal{A}_{L,L}$

9. Enumerate all pairs of vertices $c'_1 \in C_1$, $c'_2 \in C_2$.

10. Add to C all vertices x from C_1 such that $N_{L_1}(c'_1) \subseteq N_{L_1}(x)$.

11. Add to C all vertices x from C_2 such that $N_{R_2}(c'_2) \subseteq N_{R_2}(x)$.

12. Add to C every vertex $c \in L_1$ such that there exists a path $Q = c \dots v$ with the following properties: $v \in R_1$, $V(Q) \cap C = \emptyset$, $V(Q^*) \cap (V(H) \cup L_i \cup R_i) = \emptyset$ and Q^* is anticomplete to $H \setminus c_1$.

13. Add to C every vertex $c \in R_2$ such that there exists a path $Q = c \dots v$ with the following properties: $v \in L_2$, $V(Q) \cap C = \emptyset$, $V(Q^*) \cap (V(H) \cup L_2 \cup R_2) = \emptyset$ and Q^* is anticomplete to $H \setminus c_2$.

14. Check whether C is proper separator of G. If not, discard C. Otherwise return C, and go to the next step of enumeration (so step 1., 2. or 9.).

Table 2: Algorithm $\mathcal{A}_{L,R}$ (only steps 9.–14. are described)

By Lemma 4.8, we may consider two cases:

- Case 1: $R_1 \cap D$ and $R_2 \cap D$ are empty (algorithm $\mathcal{A}_{L,L}$).
- Case 2: $R_1 \cap D$ and $L_2 \cap D$ are empty (algorithm $\mathcal{A}_{L,R}$).

We shall prove that in each case, the algorithm that is indicated above outputs D. The cases being similar, we just handle Case 1, so we suppose that $R_1 \cap D$ and $R_2 \cap D$ are empty. Hence, by Lemma 4.9 there exists a vertex $c'_1 \in C_1$ such that for all $x \in C_1$, $x \in D$ if and only if $N_{L_1}(c'_1) \subseteq N_{L_1}(x)$, and there exists a similar vertex $c'_2 \in C_2$. At some point, in step 9., the vertices c'_1, c'_2 will be considered. And by Lemma 4.9, steps 10. and 11. correctly put in C the sets $C_1 \cap D$ and $C_2 \cap D$.

At this step of the algorithm, all major vertices and vertices of $C_1 \cap D$ and $C_2 \cap D$ are in C. The only vertices in $D \setminus C$ are therefore in L_1 and L_2. Let c be a vertex in $C \cap L_1$. By Lemma 4.7, there exists a viaduct with end c, so that in step 12. a path Q is detected and c is correctly added to C. Conversely, if some path Q is detected in step 12., then $c \in D$. For otherwise, since $R_1 \cap D = \emptyset$, Q is a path from L to R, so it must contain a vertex of D. Since all vertices of $D \setminus (L_1 \cup L_2)$ are in C and therefore not in Q, they are not used by Q, so there is a contradiction.

Similarly, in step 13., $D \cap L_2$ is put in C.

Now, $C = D$. Hence, in step 14., C is detected as a proper separator and the algorithm outputs D as claimed.

Complexity analysis

The enumeration of all vertices takes time $O(n^8)$, and for each of them, all the computations that we do rely on connectivity checks that can be implemented to run in time $O(n^2)$ with BFS. The total running time is therefore $O(n^{10})$. $\square$

We can now prove Theorem 2.3, restated below.

Theorem 2.3 *Every graph in $\mathcal{C}$ on n vertices contains at most $O(n^8)$ minimal separators. There is an algorithm of complexity $O(n^{10})$ that enumerates them. Consequently, there exists a polynomial time algorithm for the Maximum Weighted Independent Set restricted to $\mathcal{C}$.*

Proof. For each of the 8-tuple of vertices that is considered by algorithms $\mathcal{A}_{L,L}$ and $\mathcal{A}_{L,L}$, each algorithm outputs at most one proper separator. Hence, there is at most $O(n^8)$ such separators. As explained at the beginning of Section 4, non-proper minimal separators are all clique separators, and there are at most $O(n)$ and they can be enumerated in time $O(n^3)$. In Section 2, it is explained why this implies that the Maximum Weighted Independent Set restricted to $\mathcal{C}$ can be solved in polynomial time.

$\square$

Complexity of MWIS in $\mathcal{C}$

We do not recall here the definition of a *potential maximal clique*, see [4]. A potential maximal clique in a graph G is a subset of $V(G)$ with special properties. We denote by m the number of edges in G, by p the number of potential maximal cliques in G and by s be the number of minimal separators in G. In [4], it is proved that $p \leq O(ns^2 + ns + 1)$ (Proposition 22) and that, given the list of minimal separators, the potential maximal cliques of G can be listed in time $O(n^2ms^2)$ (Theorem 23). In [9], based on [8], it is proved that, given the list of potential maximal cliques, the MWIS problem can be solved in time $O(n^5mp)$ in any graph (Proposition 1). By Theorem 2.3, $s \leq O(n^8)$, so $p \leq O(n^{17})$. Hence, in $\mathcal{C}$, the MWIS problem can be solved in time $O(n^{24})$.

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